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# THE LIPSCHITZ CONTINUITY OF THE DISTANCE FUNCTION TO THE CUT LOCUS

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ABSTRACT. Let $N$ be a closed submanifold of a complete smooth Riemannian manifold $M$ and $U\nu$ the total space of the unit normal bundle of $N$. For each $v \in U\nu$, let $\rho(v)$ denote the distance from $N$ to the cut point of $N$ on the geodesic $\gamma_v$ with the velocity vector $\dot{\gamma}_v(0) = v$. The continuity of the function $\rho$ on $U\nu$ is well known. In this paper we prove that $\rho$ is locally Lipschitz on which $\rho$ is bounded; in particular, if $M$ and $N$ are compact, then $\rho$ is globally Lipschitz on $U\nu$. Therefore, the canonical interior metric $\delta$ may be introduced on each connected component of the cut locus of $N$, and this metric space becomes a locally compact and complete length space.

Let $N$ be an immersed submanifold of a complete $C^\infty$ Riemannian manifold $M$ and $\pi : U\nu \to N$ the unit normal bundle of $N$. For each positive integer $k$ and vector $v \in U\nu$, let a number $\lambda_k(v)$ denote the parameter value of $\gamma_v$, where $\gamma_v$ denotes the geodesic for which the velocity vector is $v$ at $t = 0$, such that $\gamma_v(\lambda_k(v))$ is the $k$-th focal point (conjugate point for the case in which $N$ is a point) of $N$ along $\gamma_v$, counted with focal (or conjugate) multiplicities. A unit speed geodesic segment $\gamma : [0, a] \to M$ emanating from $N$ is called an *$N$-segment* if $t = d(N, \gamma(t))$ on $[0, a]$. By the first variation formula, an $N$-segment is orthogonal to $N$. A point $\gamma_v(t_0)$ on an $N$-segment $\gamma_v, v \in U\nu$, is called a *cut point* of $N$ if there is no $N$-segment properly containing $\gamma[0, t_0]$. For each $v \in U\nu$, let $\rho(v)$ denote the distance from $N$ to the cut point on $\gamma_v$ of $N$. Whitehead [27] investigated the structure of the conjugate locus and the cut locus of a point on a real analytic Finsler manifold. He determined the structure of the conjugate locus around a conjugate point for which the conjugate multiplicity is locally constant on its neighborhood (cf. also [25]) and proved the continuity of the function $\rho$. In compact symmetric spaces, T. Sakai [19] and M. Takeuchi [23] determined the detailed structure of the cut locus of a point. The detailed structure of the cut locus of a point in a 2-dimensional Riemannian manifold has been investigated by Poincaré, Myers, and others [7], [11], [13]. Hartman first tried to show the absolute continuity of the function $\rho$ when $M$ is 2-dimensional. He proved in [8] that if $\rho$ is of bounded variation, then $\rho$ is absolutely continuous. Recently, Hebda [11] and the first named author [13] independently proved Ambrose's problem by showing that $\rho$ is absolutely continuous on a closed arc on which $\rho$ is bounded when $N$ is a point in a 2-dimensional Riemannian manifold. Therefore, the cut locus of a point in a compact 2-dimensional

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Riemannian manifold has finite 1-dimensional Hausdorff measure, and any two cut points can be connected by a rectifiable curve in the cut locus.

In the present paper, we prove that the focal locus and the cut locus of a submanifold of a complete $C^\infty$ Riemannian manifold have weak differentiability:

**Theorem A.** *Let $N$ be an immersed submanifold of a complete $C^\infty$ Riemannian manifold $M$ and $\pi : U\nu \to N$ the unit normal bundle of $N$. Then, for each positive integer $k$ and $v \in U\nu$ with $\lambda_k(v) < \infty$, $\lambda_k$ is locally Lipschitz around $v$.*

**Theorem B.** *Let $N$ be an embedded submanifold of a complete $C^\infty$ Riemannian manifold $M$ and $\pi : U\nu \to N$ the unit normal bundle of $N$. Then, for each $v \in U\nu$ with $\rho(v) < \infty$, $\rho$ is locally Lipschitz around $v$. In particular, if $M$ and $N$ are compact, then $\rho$ is globally Lipschitz on $U\nu$ and hence the cut locus has finite $(m-1)$-dimensional Hausdorff measure, where $m$ denotes the dimension of $M$.*

Note that $\lambda_k$ is not always differentiable (see Example 3.1). If there exists a neighborhood of $\lambda_k(v)v$ in which the focal multiplicity of each focal tangent vector is constant, then $\lambda_k$ is $C^\infty$ around $v$, as Warner [25] and Hebda [9] reported. In particular, if $M$ is 2-dimensional, then $\lambda_k$ is $C^\infty$ on which $\lambda_k$ is bounded. In fact, the focal multiplicity is 1 at each focal point.

Rademacher's theorem (cf. [16]) states that a Lipschitz map of a domain in $R^k$ into $R^l$ is differentiable almost everywhere. Therefore, as corollaries to Theorems A and B, there exist tangent spaces at almost all points in the tangent focal locus and the tangent cut locus, respectively.

Since a Lipschitz continuous function is absolutely continuous, Theorem B generalizes the previously mentioned result by Hebda and the first named author; therefore, this theorem is new, even for 2-dimensional $M$. Theorem B has a few corollaries. If a cut point $q$ is not a focal point of the submanifold along an $N$-segment, then the Hausdorff dimension of the focal locus around $q$ equals $m-1$ (cf. [14] for the case in which $N$ is a point).

**Corollary C.** *Suppose $N$ is a closed submanifold of $M$. Then the canonical interior metric $\delta$ may be introduced on each connected component of $C_N$. Moreover the topology introduced from $\delta$ coincides with the relative one of $(M, g)$, and $(C_N, \delta)$ is a locally compact and complete length space.*

Note that the cut locus of a compact subset of an Alexandrov surface admits the canonical interior metric, which is a result given by Shiohama and the second named author [22]. Corollary C raises the following interesting problem:

*Does the metric space $(C_N, \delta)$ have curvature bounded below (or above) in the sense of Alexandrov?*

The answer is no. Counterexamples are given in Section 3.

Refer to [1] or [2] for the geometry on metric spaces. [20] is a good reference on Riemannian geometry and in particular on the Morse index theorem.

Finally, the authors would like to thank Prof. T. Akamatsu for his valuable suggestions on analysis methods such as the Malgrange preparation theorem. He kindly pointed out to the second named author that if the focal multiplicity of the focal point $\gamma_v(\lambda_k(v))$ is two, then the locally Lipschitz continuity of $\lambda_k$ can be proven at $v$ by making use of the Malgrange preparation theorem (see [12]) and Lemma B.1 in [24].

## 1. The distance functions to the tangent focal locus

Let $(M, g)$ denote a complete, $m$-dimensional $C^\infty$ Riemannian manifold. We denote by $TM$ the total space of the tangent bundle over $M$, and by exp the exponential map defined on $TM$. The fiber over $p$ is denoted by $T_pM$. Let $N$ denote a $C^\infty$ $n$-dimensional submanifold of $M$ and $\pi : \nu \longrightarrow N$ the normal bundle of $N$. The fiber over $p$ is denoted by $\nu_p$. For each $\xi \in \nu_p$, let $A_\xi$ denote the shape operator of $N$ with respect to $\xi$, which is a symmetric linear transformation on $T_pN$ (see [20] for the definition of the shape operator). Suppose that a unit speed geodesic $\gamma : [0, \infty) \longrightarrow M$ is given, for which $\xi := \dot\gamma(0) \in \nu_p$. A Jacobi field $Y$ along $\gamma$ is called an *$N$-Jacobi field* if $Y$ satisfies the following two initial conditions:

$$Y(0) \in T_pN, \qquad Y'(0) - A_\xi Y(0) \in \nu_p, \tag{1.1}$$

where $Y'$ denotes the covariant derivative of the Jacobi field $Y$ along $\gamma$. Note that if $N$ consists of a single point $p$, then an $N$-Jacobi field $Y$ is a Jacobi field along $\gamma$ emanating from $p$ with $Y(0) = 0$ and $Y'(0) \in T_pM$. The following equations, (1.2) and (1.3), are very important in proving Theorems A and B (cf. [4]). For any two Jacobi fields $X, Y$ along a geodesic $\gamma : [0, \infty) \longrightarrow M$, there exists a constant $c$ such that

$$g(X'(t), Y(t)) - g(X(t), Y'(t)) = c \tag{1.2}$$

for any $t \geq 0$. In particular, the equality

$$g(X'(t), Y(t)) = g(X(t), Y'(t)) \tag{1.3}$$

holds for any $N$-Jacobi fields $X, Y$. A point $\gamma(t_0)$, where $t_0$ is a positive number (respectively $t_0\dot\gamma(0)$), is called a *focal point* (respectively *focal tangent vector*) of $N$ along a geodesic $\gamma$ emanating perpendicularly from $N$ if there exists a non-zero $N$-Jacobi field $Y$ along $\gamma$ with $Y(t_0) = 0$. For each geodesic $\gamma : [0, b] \longrightarrow M$ emanating perpendicularly from $N$, let $\mathrm{ind}_N(\gamma)$ denote the index of $\gamma$ (see [20] for the definition of the index). Let $\pi : U\nu \longrightarrow N$ denote the unit sphere normal bundle over $N$. For each positive integer $k$ and each unit tangent vector $v \in U\nu$ we define a number $\lambda_k(v)$ by

$$\lambda_k(v) := \sup\{t; \mathrm{ind}_N(\gamma_v|_{[0,t]}) \leq k - 1\}, \tag{1.4}$$

where $\gamma_v$ denotes the geodesic $\gamma_v(t) := \exp(tv)$. The differential of the normal exponential map $\exp^\perp$ is singular at $v \in \nu$ if and only if $\exp(v)$ is a focal point of $N$ along $\gamma_v$. It is clear that $0 < \lambda_1(v) \leq \lambda_2(v) \leq \lambda_3(v) \leq \cdots$ and it follows from the Morse index theorem (cf. [20], also [15] or [16]) that $\gamma_v(\lambda_k(v))$ is the $k$-th focal point of $N$ along $\gamma_v$, counted with focal multiplicities. Here the *focal multiplicity* of a focal point $\gamma_v(t_0)$ is the dimension of the kernel of $d\exp^\perp$ at $t_0 v$, where $d\exp^\perp$ denotes the differential of $\exp^\perp$. Hence $\lambda_k(v)$ is the distance function to the $k$-th focal tangent vector of $N$ along $\gamma_v$, counting focal multiplicities.

**Definition 1.1.** For each $v \in U\nu$ and $w \in T_{\pi(v)}M$, let $Y(t; v, w)$ denote the $N$-Jacobi field $Y(t)$ along the geodesic $\gamma_v$ with initial conditions $Y(0) = w^T$ and $Y'(0) = A_v w^T + w^\perp$, where $w^T$ and $w^\perp$ denote the images of $w$ under orthogonal projection to $T_{\pi(v)}N$ and $\nu_{\pi(v)}$, respectively.

**Definition 1.2.** For each positive integer $k$ and $v \in U_p\nu := \nu_p \cap U\nu$ with $\lambda_k(v) < \infty$, let $F(\lambda_k(v)v)$ denote the kernel of the linear map $w \in T_pM \longrightarrow Y(\lambda_k(v); v, w) \in T_{\gamma_v(\lambda_k(v))}M$.

Note that the dimension of $F(\lambda_k(v)v)$ is the same as the focal multiplicity of the focal point $\gamma_v(\lambda_k(v)v)$.

**Lemma 1.1.** *Let $\{v_j\}$ be a sequence of vectors in $U\nu$ convergent to a tangent vector $v \in U_p\nu$. Suppose that there exist positive integers $k_1, \cdots, k_l$ such that the sequences $\{\lambda_{k_i}(v_j)\}_j$ converge to a common real number $t_0$, and that $\lambda_{k_1}(v_j) < \lambda_{k_2}(v_j) < \cdots < \lambda_{k_l}(v_j)$ for each $j$. If there exists a linear subspace $F_i := \lim_{j\to\infty} F(\lambda_{k_i}(v_j)v_j)$ of $F(t_0v)$ for each $i = 1, \cdots l$, i.e., there exists a convergent sequence of a basis of $F(\lambda_{k_i}(v_j)v_j)$, then $Y'(t_0; v, x)$ and $Y'(t_0; v, y)$ are orthogonal for any $x \in F_a$ and $y \in F_b$ $(a < b)$, and in particular the dimension of $F_1 + \cdots + F_l$ equals $\sum_{i=1}^{l} \dim F_i$.*

*Proof.* Let $\{x_j\}$ and $\{y_j\}$ be sequences of elements of $F(\lambda_{k_a}(v_j)v_j)$ and $F(\lambda_{k_b}(v_j)v_j)$ convergent to $x$ and $y$ respectively. Then, from (1.3) it follows that

$$g(Y'(t; v_j, x_j), Y(t; v_j, y_j)) = g(Y(t; v_j, x_j), Y'(t; v_j, y_j))$$

for any $t \geq 0$. Since $Y(\lambda_{k_a}(v_j); v_j, x_j) = 0$, we get

$$g(Y'(\lambda_{k_a}(v_j); v_j, x_j), Y(\lambda_{k_a}(v_j); v_j, y_j)) = 0. \tag{1.5}$$

Since $Y(t; v_j, y_j) = 0$ at $t = \lambda_{k_b}(v_j)$, there exists a $C^\infty$ vector field $X(t; v_j, y_j)$ along $\gamma_{v_j}$ that is smoothly dependent on $(v_j, y_j)$ and such that

(1.6)
$$Y(t; v_j, y_j) = (t - \lambda_{k_b}(v_j))X(t; v_j, y_j), \quad X(\lambda_{k_b}(v_j); v_j, y_j) = Y'(\lambda_{k_b}(v_j); v_j, y_j).$$

By (1.5) and (1.6), we get

$$g(Y'(\lambda_{k_a}(v_j); v_j, x_j), X(\lambda_{k_a}(v_j); v_j, y_j)) = 0. \tag{1.7}$$

If we take the limit of (1.7), then it follows from (1.6) that

$$g(Y'(t_0; v, x), Y'(t_0; v, y)) = 0. \tag{1.8}$$

Let $f$ denote the linear map of $T_pM$ into $T_{\gamma_v(t_0)}M$ defined by $f(w) = Y'(t_0; v, w)$. Since the $f(F_i)$, $i = 1, \cdots, l$, are mutually orthogonal by (1.8), we have

$$\sum_{j=1}^{l} \dim f(F_i) = \dim(f(F_1) + \cdots + f(F_l)) \leq \dim(F_1 + \cdots + F_l).$$

Since $f|_{F_i}$ is injective, $\dim f(F_i) = \dim F_i$ for each $i$. Therefore, the dimension of $F_1 + \cdots + F_l$ equals $\sum_{i=1}^{l} \dim F_i$. □

**Proposition 1.2.** *For each positive number $t$, the function*

$$v \in U\nu \longrightarrow \mathrm{ind}_N(\gamma_v|_{[0,t]})$$

*is locally constant around each tangent vector $v \in U\nu$ if $\gamma_v(t)$ is not a focal point of $N$ along $\gamma_v$. Furthermore, the function $\lambda_k : U\nu \longrightarrow (0, \infty]$ is continuous for each $k$.*

*Proof.* Take a vector $v_0 \in U\nu$ such that $\gamma_{v_0}(t)$ is not a focal point of $N$ along $\gamma_{v_0}$. Since the index form depends continuously on the geodesic segment $\gamma_v|_{[0,t]}$, it is clear that

$$\mathrm{ind}_N(\gamma_{v_0}|_{[0,t]}) \le \mathrm{ind}_N(\gamma_v|_{[0,t]}) \tag{1.9}$$

for any $v \in U\nu$ sufficiently close to $v_0$. Suppose that there exists a sequence $\{v_j\}$ of elements of $U\nu$ convergent to $v_0$ such that $\mathrm{ind}_N(\gamma_{v_0}|_{[0,t]}) \neq \mathrm{ind}_N(\gamma_{v_j}|_{[0,t]})$. By taking a subsequence of the sequence, and by (1.9), we may assume that

$$\mathrm{ind}_N(\gamma_{v_0}|_{[0,t]}) < \mathrm{ind}_N(\gamma_{v_j}|_{[0,t]}) \tag{1.10}$$

for any $j$, and that the limit linear space $F_k := \lim_{j\to\infty} F(\lambda_k(v_j)v_j)$ exists for each $k$ with $\lim_{j\to\infty}\lambda_k(v_j) < t$. It follows from the Morse index theorem and (1.11) that

$$\mathrm{ind}_N(\gamma_{v_j}|_{[0,t]}) = \sum \dim F(\lambda_k(v_j)v_j) = \sum \dim F_k \tag{1.11}$$

for any sufficiently large $j$, where the sums are taken over the set $\{\lambda_k(v_j); \lambda_k(v_j) < t\}$. It follows from the Morse index theorem and Lemma 1.1 that

$$\mathrm{ind}_N(\gamma_{v_0}|_{[0,t]}) \ge \sum \dim F_k = \mathrm{ind}_N(\gamma_{v_j}|_{[0,t]}). \tag{1.12}$$

However, a contradiction exists between (1.10) and (1.12). Therefore, the function $v \in U\nu \longrightarrow \mathrm{ind}_N(\gamma_v|_{[0,t]})$ is locally constant around each tangent vector $v \in U\nu$ if $\gamma_v(t)$ is not a focal point of $N$ along $\gamma_v$. Take any $v_0 \in U\nu$ and any positive number $t > \lambda_k(v_0)$ (respectively $t < \lambda_k(v_0)$) such that $\gamma_{v_0}(t)$ is not a focal point of $N$ along $\gamma_{v_0}$. Since $\mathrm{ind}_N(\gamma_v|_{[0,t]})$ is locally constant around $v_0$, we get $\lambda_k(v) > t$ (respectively $\lambda_k(v) < t$) for any $v$ sufficiently close to $v_0$, implying the continuity of $\lambda_k$. □

Fix any positive integer $k$ and any $v_0 \in U_p\nu$ with $\lambda_k(v_0) < +\infty$. We want to prove the local Lipschitz continuity of $\lambda_k$ around $v_0$. For convenience, introduce a $C^\infty$ Riemannian metric $G$ on $U\nu$. The Riemannian distance function induced from $G$ is denoted by $D$. For each positive number $\delta$, we denote the open ball centered at $v_0$ with radius $\delta$ by $B_D(v_0;\delta)$.

**Definition 1.3.** For each $q \in M$, let $S_qM$ denote the set of all unit tangent vectors of $T_qM$, and for each tangent vector $v$, let $\|v\|$ denote the length of $v$, i.e., $\|v\| := \sqrt{g(v,v)}$.

Since $\lambda_k$ is continuous, there exists a relatively compact convex neighborhood $B_D(v_0;\delta_0(k))$, on which $\lambda_k$ does not exceed $\lambda_k(v_0)+1$. Since each Jacobi field $Y(t)$ is uniquely determined by $Y(t_1)$ and $Y'(t_1)$ for some $t_1$, the number

(1.13)

$$2C_0(J',k) := \min\{\ \|Y'(\lambda_i(v_0); v_0, w)\|^2\ ;\ 1 \le i \le k,\ w \in S_pM \cap F(\lambda_i(v_0)v_0)\ \}$$

is positive. Since each $\lambda_i$ is continuous, there exists a positive number $\delta_1(k)$ ($\le \delta_0(k)$) such that

$$C_0(J',k) \le \|Y'(\lambda_i(v); v, w)\|^2 \tag{1.14}$$

for any $v \in B_D(v_0;\delta_1(k))$ and any $w \in F(\lambda_i(v)v) \cap S_{\pi(v)}M,\ 1 \le i \le k$. For each $v \in B_D(v_0;\delta_1(k))$, choose a sufficiently small positive number $\epsilon(v)$ with the following two properties: The closed intervals $[s_i(v), t_i(v)],\ 1 \le i \le k$, are mutually disjoint if $\lambda_i(v) \neq \lambda_j(v)$, where $s_i(v) := \lambda_i(v) - \epsilon(v),\ t_i(v) := \lambda_i(v) + \epsilon(v)$. For each positive integer $i (\le k)$, the geodesic segment $\gamma_v|_{[s_i(v), t_i(v)]}$ lies in a convex ball.

**Definition 1.4.** For each $v \in B_D(v_0; \delta_1(k)), \tau \in (\lambda_i(v), t_i(v)]$, and $w \in F(\lambda_i(v)v)$ $(1 \le i \le k)$, let $X(t; v, w, \tau)$ denote the broken Jacobi field $X(t)$ along $\gamma_v$ such that

$$X(t) = \begin{cases} Y(t; v, w) & \text{on } [0, s_i(v)], \\ Y(t; v, w, \tau) & \text{on } [s_i(v), \tau], \\ 0 & \text{on } [\tau, \infty], \end{cases}$$

where $Y(t; v, w, \tau)$ denotes the Jacobi field along $\gamma_v$ satisfying

$$Y(s_i(v); v, w, \tau) = Y(s_i(v); v, w), \qquad Y(\tau; v, w, \tau) = 0.$$

Note that the Jacobi field $Y(t; v, w, \tau)$ is uniquely determined by the property

$$Y(\tau; v, w, \tau) = 0, \qquad Y(s_i(v); v, w, \tau) = Y(s_i(v); v, w)$$

for each $\tau \in (s_i(v), t_i(v)]$, since $\gamma_v|_{[s_i(v), t_i(v)]}$ lies in a convex ball. The uniqueness implies that $Y(t; v, \sum_j c_j w_j, \tau) = \sum_j c_j Y(t; v, w_j, \tau)$, and thus

$$X(t; v, \sum_j c_j w_j, \tau) = \sum_j c_j X(t; v, w_j, \tau)$$

for any finitely many real numbers $c_j$ and vectors $w_j$ which are elements in a common $F(\lambda_i(v)v)$. By taking a smaller $\epsilon(v)$, we may assume that the length $\|X(t; v, w, \tau)\|$ of $X(t; v, w, \tau)$ is monotone on $[s_i(v), \tau]$. Therefore, if

$$\begin{aligned} C(J, k) := \sup\{ \|Y(t; v, w)\|^2 \, ; \, 0 \le t \le \lambda_k(v_0) + 1, \\ v \in B_D(v_0; \delta_1(k)), \ w \in S_{\pi(v)}M\}, \end{aligned} \tag{1.15}$$

then

$$C(J, k) \ge \|X(t; v, w, \tau)\|^2 \tag{1.16}$$

on $[0, \infty)$ for each broken Jacobi field $X(t; v, w, \tau)$. Let $\{e_1, \cdots, e_m\}$ denote a $C^\infty$ local frame field on a neighborhood $V$ of $p = \pi(v_0)$ such that $\{e_1(q), \cdots, e_m(q)\}$ and $\{e_1(q), \cdots, e_n(q)\}$ are orthonormal bases of $T_qM$ and $T_qN$ for each $q \in N \cap V$, respectively.

**Definition 1.5.** For each $v \in U\nu \cap \pi^{-1}(V \cap N)$ let $\{E_1(t; v), \cdots, E_m(t; v)\}$ denote the set of parallel vector fields along the geodesic $\gamma_v$ such that $E_i(0; v) = e_i(\pi(v))$ for each $i$.

Choose a positive number $\delta_2(k)$ $(\le \delta_1(k))$ so as to satisfy

$$B_D(v_0; \delta_2(k)) \subset U\nu \cap \pi^{-1}(V \cap N).$$

Let $I_0^t$ denote the index form with respect to a geodesic $\gamma_v|_{[0,t]}$, i.e.,

$$\begin{aligned} I_0^t(X, Y) = \int_0^t g(X'(t), Y'(t)) - g(R(X(t), \dot\gamma_v(t))\dot\gamma_v(t), Y(t))dt \\ + g(A_v(X(0)), Y(0)) \end{aligned}$$

for piecewise $C^\infty$ vector fields $X, Y$ along $\gamma_v|_{[0,t]}$, where $R$ denotes the sectional curvature tensor field of $(M, g)$. For simplicity, $I_0^t(X, X)$ will be denoted by $I_0^t(X)$. Since

$$R_{ij}(t, v) := g(R(E_i(t; v), \dot\gamma_v(t))\dot\gamma_v(t), E_j(t; v)), \qquad i, j = 1, \ldots, m,$$

$$f_{kl}(v) := g(A_v(e_k(\pi(v))), e_l(\pi(v))), \qquad k, l = 1, \ldots, n,$$

are $C^\infty$ functions, we may choose constants $C(R,k)$ and $C(A)$ such that the inequalities

$$
\begin{aligned}
|R_{ij}(t,v_1) - R_{ij}(t,v_2)| &\le C(R,k)D(v_1,v_2),\\
|f_{kl}(v_1) - f_{kl}(v_2)| &\le C(A)D(v_1,v_2)
\end{aligned} \tag{1.17}
$$

hold for any $t \in [0, \lambda_k(v_0)+1]$, $i,j \in \{1,\dots,m\}$, $k,l \in \{1,\dots,n\}$ and $v_1, v_2 \in B_D(v_0;\delta_3(k))$, where $\delta_3(k) := \frac{1}{2}\delta_2(k)$.

**Lemma 1.3.** *For any $v \in B_D(v_0;\delta_3(k)), w \in F(\lambda_i(v)v)$ and $\tau \in (\lambda_i(v), t_i(v)]$ $(1 \le i \le k)$,*

$$
I_0^\tau(X(\cdot\,;v,w,\tau)) = -g(Y(\tau;v,w), Y'(\tau\,;v,w,\tau)). \tag{1.18}
$$

*Moreover, for each $v \in B_D(v_0;\delta_3(k))$ and positive integer $i(\le k)$, there exists a real number $\tau_i(v) \in (\lambda_i(v), t_i(v))$ such that, for any $\tau \in (\lambda_i(v), \tau_i(v))$ and $w \in F(\lambda_i(v)v)$,*

$$
I_0^\tau(X(\cdot\,;v,w,\tau)) \le -\frac{1}{2}C_0(J',k)(\tau - \lambda_i(v))\|w\|^2. \tag{1.19}
$$

*Proof.* Since $X(t;v,w,\tau)|_{[0,s_i(v)]}$ and $X(t;v,w,\tau)|_{[s_i(v),\tau]}$ are Jacobi fields along $\gamma_v$, we get

$$
\begin{aligned}
I_0^\tau(X(\cdot\,;v,w,\tau)) = g(&Y'(s_i(v);v,w), Y(s_i(v);v,w,\tau))\\
&- g(Y(s_i(v);v,w), Y'(s_i(v);v,w,\tau)).
\end{aligned}
$$

It follows from (1.2) that

$$
I_0^\tau(X(\cdot\,;v,w,\tau)) = g(Y'(\tau;v,w), Y(\tau;v,w,\tau)) - g(Y(\tau;v,w), Y'(\tau;v,w,\tau)).
$$

Since $Y(\tau;v,w,\tau) = 0$, equation (1.18) holds. Since $Y(t;v,w) = 0$ at $t = \lambda_i(v)$, there exists a $C^\infty$ vector field $X(t;v,w)$ such that $Y(t;v,w) = (t - \lambda_i(v))X(t;v,w)$. Since

$$
\lim_{\tau\to\lambda_i(v)} Y'(\tau;v,w,\tau) = Y'(\lambda_i(v);v,w,\lambda_i(v)) = Y'(\lambda_i(v);v,w) = \lim_{\tau\to\lambda_i(v)} X(\tau;v,w),
$$

it follows from (1.14) that there exists $\tau_i(v) \in (\lambda_i(v), t_i(v))$ such that

$$
\begin{aligned}
-g(Y(\tau;v,w), Y'(\tau;v,w,\tau)) &\le -\frac{1}{2}\|Y'(\lambda_i(v);v,w)\|^2(\tau - \lambda_i(v))\\
&\le -\frac{1}{2}C_0(J',k)\|w\|^2(\tau - \lambda_i(v))
\end{aligned}
$$

for any $\tau \in (\lambda_i(v), \tau_i(v))$ and $w \in F(\lambda_i(v)v)$, completing the proof of (1.19). $\square$

*Proof of Theorem A.* Fix any $v_1 \in B_D(v_0;\delta_3(k))$. We prove that the inequality

$$
\lambda_k(v_2) - \lambda_k(v_1) \le L_k D(v_1,v_2)
$$

holds for any $v_2 \in B_D(v_0;\delta_3(k))$ sufficiently close to $v_1$, where

$$
L_k := \frac{4mkC(J,k)}{C_0(J',k)}(C(A) + (\lambda_k(v_0)+1)C(R,k)).
$$

Thus, the above inequality can be easily proven for any $v_2 \in B_D(v_0;\delta_3(k))$, and $\lambda_k$ is Lipschitz continuous on $B_D(v_0;\delta_3(k))$ with Lipschitz constant $L_k$. For each positive integer $i \le k$, choose a unit vector $w_i$ from $F(\lambda_i(v_1)v_1)$ so as to satisfy the following property: for distinct $i,j \le k$, $w_i$ and $w_j$ are orthogonal whenever $\lambda_i(v_1) = \lambda_j(v_1)$. Set $a_i := \lambda_i(v_1) + \epsilon$, where $\epsilon$ $(\le 1)$ is a sufficiently small positive number satisfying $a_i \in (\lambda_i(v_1), \tau_i(v_1))$ for each $i \le k$. Let $W(\gamma_{v_1})$ denote the $k$-dimensional linear space spanned by piecewise $C^\infty$ vector fields $X_i(t; v_1),\ 1 \le i \le k$, along $\gamma_{v_1}$, where $X_i(t; v_1) := X(t; v_1, w_i, a_i)$. We first prove that the inequality

$$I_0^{a_k}\Big(\sum_{i=1}^{k} c_i X_i(\cdot\,; v_1)\Big) \le -\frac{\epsilon}{2} C_0(J', k) \sum_{i=1}^{k} c_i^2 \tag{1.20}$$

holds for any real numbers $c_i$'s. Choose a maximal subset $\{i_1, ..., i_l\}$ of $\{1, ..., k\}$ satisfying $\lambda_{i_1}(v_1) < \lambda_{i_2}(v_1) < \cdots < \lambda_{i_l}(v_1)$. Set

$$N_s := \{j; \lambda_j(v_1) = \lambda_{i_s}(v_1)\}$$

for each $s \in \{1, ..., l\}$. The fact that the $N_s$ are mutually disjoint subsets of $\{1, \cdots, k\}$ with $N_1 \cup \cdots \cup N_l = \{1, \cdots, k\}$ is trivial. Since

$$\sum_{i=1}^{k} c_i X_i(t; v_1) = \sum_{s=1}^{l} X(t; v_1, \sum_{i \in N_s} c_i w_i, a_{i_s}),$$

it follows that

$$I_0^{a_k}\Big(\sum_{i=1}^{k} c_i X_i(\cdot\,; v_1)\Big) = \sum_{s=1}^{l} I_0^{a_k}\Big(X(\cdot\,; v_1, \sum_{i \in N_s} c_i w_i, a_{i_s})\Big). \tag{1.21}$$

Note that

$$I_0^{a_k}(X(\cdot\,; v_1, x_i, a_i), X(\cdot\,; v_1, x_j, a_j)) = 0$$

for any $x_i \in F(\lambda_i(v_1)v_1), y_i \in F(\lambda_j(v_1)v_1)$ with $\lambda_i(v_1) < \lambda_j(v_1)$. By applying (1.19) to each broken Jacobi field $X(t; v_1, \sum_{i \in N_s} c_i w_i, a_{i_s})$, it follows that (1.21) implies (1.20). Choose $v_2 \in U(v_0; \delta_3(k))$ sufficiently close to $v_1$ to satisfy

$$\epsilon := L_k D(v_1, v_2) < \min\{\tau_i(v_1) - \lambda_i(v_1); 1 \le i \le k\}.$$

By (1.20), the inequality

$$I_0^{a_k}\Big(\sum_{i=1}^{k} c_i X_i(\cdot\,; v_1)\Big) \le -\frac{L_k}{2} C_0(J', k) D(v_1, v_2) \sum_{i=1}^{k} c_i^2 \tag{1.22}$$

holds for any $v_2 \in B_D(v_0; \delta_3(k))$ sufficiently close to $v_1$ and any real numbers $c_i$. For each $X \in W(\gamma_{v_1})$, we construct a piecewise $C^\infty$ vector field $\tilde{X}(t)$ along $\gamma_{v_2}$ by

$$\tilde{X}(t) := \sum_{i=1}^{m} g(X(t), E_i(t; v_1)) E_i(t; v_2).$$

For simplicity, set

$$Z(t) := \sum_{i=1}^{k} c_i X_i(t; v_1).$$

It follows from (1.17) and the Schwarz inequality that

$$I_0^{a_k}(\tilde{Z}) \le I_0^{a_k}(Z) + mkC(J, k)D(v_1, v_2)(C(A) + (\lambda_k(v_0) + 1)C(R, k)) \sum_{i=1}^{k} c_i^2.$$

Hence, by (1.22), we get

$$I_0^{a_k}(\tilde{Z}) \le -\frac{1}{4} L_k C_0(J', k) D(v_1, v_2) \sum_{i=1}^{k} c_i^2,$$

which holds for any $v_2 \in B_D(v_0; \delta_3(k))$ sufficiently close to $v_1$. This inequality implies the index form $I_0^{a_k}$ is negative definite on the $k$-dimensional linear space $\{\tilde{X}(t); X \in W(\gamma_{v_1})\}$, and so $\mathrm{ind}_N(\gamma_{v_2}|_{[0,a_k]})$ is not less than $k$. Therefore,

$$\lambda_k(v_2) \leq a_k = \lambda_k(v_1) + L_k D(v_1, v_2)$$

for any $v_2 \in D(v_0; \delta_3(k))$ sufficiently close to $v_1$, completing the proof of Theorem A. ☐

## 2. The distance function to the cut locus

Throughout this section $N$ always denotes an embedded submanifold of $M$. A unit speed geodesic segment $\gamma : [0, a] \to M$ emanating from $N$ is called an $N$-*segment* if $t = d(N, \gamma(t))$ on $[0, a]$. Note that any $N$-segment is orthogonal to $N$, a consequence of the first variation formula.

**Definition 2.1.** For each point $x \in M \setminus N$,

$$\Lambda_N(x) := \{-\dot{\gamma}(d(N, x))\,; \gamma \textit{ is an } N\textit{-segment reaching } x\}.$$

**Definition 2.2.** For any distinct points $x, y$ lying in a convex neighborhood around $x$, we define a unit tangent vector $v_x(y)$ at $x$ by

$$v_x(y) := \dot{\gamma}(0),$$

where $\gamma : [0, b] \to M$ denotes the unique unit speed minimizing geodesic joining $x$ to $y$.

**Lemma 2.1.** *Let $\{x_n\}$ be a sequence of points in $M \setminus N$ converging to a point $x \notin N$. For each $x_n$, choose an element $w_n$ in $\Lambda_N(x_n)$. If $\lim_{n\to\infty} v_x(x_n) =: v$ and $\lim_{n\to\infty} w_n =: w_\infty \in \Lambda_N(x)$ exist, then*

$$\angle(v, w_\infty) = \min\{\angle(v, w); w \in \Lambda_N(x)\},$$

*where $\angle(v, w_\infty)$ denotes the angle made by $v$ and $w_\infty$. Moreover,*

$$\lim_{n\to\infty} \frac{d(N, x_n) - d(N, x)}{d(x_n, x)} = -\cos\angle(v, w_\infty).$$

*Remark.* This lemma holds even when $N$ is a point in an Alexandrov space; cf. Lemma 6.3 in [21] and Theorem 3.5 in [18].

*Proof.* Define $N$-segments $\alpha$ and $\alpha_n$ by

$$\alpha(t) := \exp((t - d(N, x))w_\infty), \qquad \alpha_n(t) := \exp((t - d(N, x))w_n).$$

Fix any $N$-segment $\beta$ reaching $x$ and choose a point $y$ ($\neq x$) on $\beta$ in a convex neighborhood $V_x$ around $x$. Let $\eta$ denote the angle made by $v$ and $w := -\dot{\beta}(d(N, x))$. It follows from the first variation formula that there exists a constant $C$ such that

$$d(y, x_n) - d(y, x) \leq -d(x_n, x)\cos\eta_n + C\, d(x_n, x)^2$$

for any sufficiently large $n$, where $\eta_n = \angle(v_x(x_n), w)$. By the triangle inequality,

$$d(N, x_n) - d(N, x) \leq d(y, x_n) - d(y, x)$$

for any $n$. Thus, we get

$$\limsup_{n\to\infty} \frac{d(N, x_n) - d(N, x)}{d(x_n, x)} \leq -\lim_{n\to\infty} \cos\eta_n = -\cos\eta. \tag{2.1}$$

On the other hand, choose a point $z(\neq x)$ on $\alpha$ in the neighborhood $V_x$. For each $n$, choose a point $y_n$ lying on $\alpha_n$ satisfying $d(y_n, x_n) = d(x, z)$. Hence, the sequence $\{y_n\}$ converges to $z$. By the triangle inequality,

$$d(N, x_n) - d(N, x) \geq d(y_n, x_n) - d(y_n, x)$$

for any $n$. Let $\theta_n$ denote the angle made by $v_x(x_n)$ and $v_x(y_n)$. By the hypothesis, the sequence $\{\theta_n\}$ converges to $\angle(v, w_\infty)$. Since the distance function is $C^\infty$ around $(x, z)$, it follows from the first variation formula that there exists a positive constant $C$ such that

$$d(y_n, x_n) - d(y_n, x) \geq -d(x_n, x)\cos\theta_n - C\, d(x_n, x)^2$$

for any sufficiently large $n$. Thus,

$$\liminf_{n\to\infty} \frac{d(N, x_n) - d(N, x)}{d(x_n, x)} \geq -\lim_{n\to\infty} \cos\theta_n = -\cos\angle(v, w_\infty). \tag{2.2}$$

By (2.1) and (2.2), we complete the proof. $\square$

**Definition 2.3.** We define a function $\rho(v)$, $v \in U\nu$, which is called *the distance function to the cut locus* of $N$, by

$$\rho(v) := \sup\{t; \gamma|_{[0,t]} \text{ is an } N\text{-segment}\}.$$

The set

$$C_N := \{\gamma_v(\rho(v)); v \in U\nu, \rho(v) < \infty\}$$

is called the *cut locus* of $N$, and each point of $C_N$ is called a *cut point* of $N$.

Note that $\rho$ is positive on $U\nu$, since $N$ is an embedded submanifold of $M$. It is well-known that $\rho$ is continuous and $\rho \leq \lambda_1$ on $U\nu$ (for example, see [20]). Let $v : (a, b) \to (U\nu, G)$ denote a unit speed geodesic on $U\nu$, where $G$ is a $C^\infty$ Riemannian metric on $U\nu$, assuming that

$$\rho(s) := \rho(v(s)) < \lambda(s) := \lambda_1(v(s))$$

on $(a, b)$.

**Definition 2.4.** For each $v(s)$ define an $N$-Jacobi field $Y_N(t; v(s))$ along $\gamma_{v(s)}$ by

$$Y_N(t; v(s)) := \frac{\partial}{\partial s} \exp(t\, v(s)).$$

Actually, $Y_N(t; v(s))$ is a Jacobi field satisfying the initial conditions

$$Y_N(0; v(s)) = d\pi(\dot{v}(s)), \qquad Y_N'(0; v(s)) = A_{v(s)}(d\pi(\dot{v}(s))) + (v'(s))^\perp.$$

**Definition 2.5.** For each $s \in (a, b)$ we define the unit tangent vectors $e_1(s)$ and $e_2(s)$ by

$$e_1(s) := -\dot{\gamma}_{v(s)}(\rho(s)), \qquad e_2(s) := \frac{1}{||Y_N(\rho(s); v(s))||} Y_N(\rho(s); v(s)).$$

Note that $e_1(s)$ and $e_2(s)$ are mutually orthogonal according to (1.2). Since we assumed $\rho < \lambda$ on $(a, b)$, the continuous curve $c(s) := \exp(\rho(s)v(s))$ lies in an immersed surface

$$S := \{\exp(t\, v(s)); s \in (a, b),\ 0 < t < \lambda(s)\}$$

of $M$. It is clear that $\{e_1(s), e_2(s)\}$ is an orthonormal basis for the tangent space $T_{c(s)}S$ for each $s \in (a, b)$. For each $w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}$, let $H(w)$ denote the hypersurface of $T_{c(s)}M$ orthogonal to $w - e_1(s)$. The dimension of the linear space $T_{c(s)}S \cap H(w)$ is 1, since $e_1(s)$ is tangent to $S$, but not to $H(w)$. Therefore, for each $w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}$ there exists a unique unit tangent vector $\eta_+(w)$ (respectively $\eta_-(w)$) in $T_{c(s)}S \cap H(w)$ such that the angle made by $\eta_+(w)$ (respectively $\eta_-(w)$) and $e_2(s)$ is smaller (respectively greater) than $\frac{\pi}{2}$.

**Definition 2.6.** For each $s \in (a, b)$, let $\xi_+(s)$ (respectively $\xi_-(s)$) denote the unique element $\eta_+(w_+(s))$ (resp. $\eta_-(w_-(s))$ in $\{\eta_+(w); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}$ (resp. $\{\eta_-(w); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}$) such that

$$\angle(\eta_+(w_+(s)), e_1(s)) = \min\{\angle(\eta_+(w), e_1(s)); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}$$

or, respectively,

$$\angle(\eta_-(w_-(s)), e_1(s)) = \min\{\angle(\eta_-(w), e_1(s)); w \in \Lambda_N(c(s)) \setminus \{e_1(s)\}\}.$$

Note that the choices of $w_\pm(s)$ may not be unique. Choose one $w_\pm(s)$ corresponding to each $s \in (a, b)$, and fix them.

**Proposition 2.2.** *At each $s_0 \in (a, b)$,*

$$\lim_{s \to s_0+0} v_{c(s_0)}(c(s)) = \xi_+(s_0) \tag{2.3}$$

*and*

$$\lim_{s \to s_0-0} v_{c(s_0)}(c(s)) = \xi_-(s_0). \tag{2.4}$$

*Furthermore, the right and left derivatives $D^+\rho(s_0)$ and $D^-\rho(s_0)$ of $\rho$ exist, and*

$$D^+\rho(s_0) = -||Y_N(\rho(s_0); v(s_0))|| \cot \theta_+(s_0) \tag{2.5}$$

*and*

$$D^-\rho(s_0) = -||Y_N(\rho(s_0); v(s_0))|| \cot \theta_-(s_0), \tag{2.6}$$

*where*

$$\theta_+(s_0) := \angle(\xi_+(s_0), e_1(s_0)), \quad \theta_-(s_0) := \angle(\xi_-(s_0), -e_1(s_0)).$$

*Proof.* Only equations (2.3) and (2.5) are proven, because the other equations may be proven in the same manner. Let $\{s_i\}$ denote a monotone decreasing sequence converging to $s_0$ such that $\eta(s_0) := \lim_{i\to\infty} v_{c(s_0)}(c(s_i))$ exists. By applying Lemma 2.1 to the sequences $\{c(s_i)\}_i$ and $\{e_1(s_i)\}_i$, we have

$$\angle(e_1(s_0), \eta(s_0)) = \min\{\angle(w, \eta(s_0)); w \in \Lambda_N(c(s_0))\}. \tag{2.7}$$

On the other hand, there exists a unit tangent vector $w \in \Lambda_N(c(s_0)) \setminus \{e_1(s_0)\}$ that is a limit vector of a sequence $\{w_i\}$, where $w_i \in \Lambda_N(c(s_i)) \setminus \{e_1(s_i)\}$, because $c(s_0)$ is not a focal point of $N$. Thus it follows from Lemma 2.1 that

$$\angle(w, \eta(s_0)) = \min\{\angle(w, \eta(s_0)); w \in \Lambda_N(c(s_0))\}. \tag{2.8}$$

By (2.7) and (2.8), $\eta(s_0)$ is a unit tangent vector in $H(w) \cap T_{c(s_0)}S$ such that $\angle(\eta(s_0), e_2(s_0))$ is smaller than $\frac{\pi}{2}$, and hence equals $\xi_+(s_0)$. This implies that $\lim_{s\to s_0+0} v_{c(s_0)}(c(s))$ exists and is equal to $\xi_+(s_0)$. It follows from Lemma 2.1 that

$$\lim_{s \to s_0+0} \frac{\rho(s) - \rho(s_0)}{d(c(s), c(s_0))} = -\cos \theta_+(s_0). \tag{2.9}$$

For each $s > s_0$ sufficiently close to $s_0$, choose the nearest point $\gamma_{v(s)}(a(s))$ on $\gamma_{v(s)}|_{[\rho(s_0)-\delta, \rho(s_0)+\delta]}$ to the point $c(s_0)$, where $\delta$ is a sufficiently small positive number such that $\gamma_{v(s)}[\rho(s_0) - \delta, \rho(s_0) + \delta]$ lies in a convex neighborhood around $c(s_0)$.

So we may assume that $\gamma_{v(s)}$ is orthogonal at $\gamma_{v(s)}(a(s))$ to the minimal geodesic joining $\gamma_{v(s)}(a(s))$ to $c(s_0)$ for each $s > s_0$ sufficiently close to $s_0$. Thus, we have

$$\lim_{s\to s_0+0} \frac{k(s)}{d(c(s_0), c(s))} = \sin\theta_+(s_0), \tag{2.10}$$

where $k(s) := d(c(s_0), \gamma_{v(s)}(a(s)))$. Since $\lim_{s\to s_0+0} v_{c(s_0)}(\gamma_{v(s)}(a(s))) = e_2(s_0)$ is perpendicular to $e_1(s_0)$, it follows from Lemma 2.1 that

$$\lim_{s\to s_0+0} \frac{a(s) - \rho(s_0)}{k(s)} = 0. \tag{2.11}$$

By the triangle inequality, we have

$$-|a(s) - \rho(s_0)| + m(s) \le k(s) \le m(s),$$

which holds for each $s > s_0$ sufficiently close to $s_0$, where

$$m(s) := d(c(s_0), \gamma_{v(s)}(\rho(s_0))).$$

Therefore, by (2.11), we get the equality

$$\lim_{s\to s_0+0} \frac{m(s)}{k(s)} = 1. \tag{2.12}$$

Let $\exp^{-1}_{c(s_0)}$ denote the local inverse of $\exp_{c(s_0)} := \exp|_{T_{c(s_0)}M}$ around $c(s_0)$. Since $d\exp_{c(s_0)}$ is the identity map on $T_{c(s_0)}M$ at the zero vector, it follows that

$$\lim_{s\to s_0+0} \frac{m(s)}{s - s_0} = ||\frac{\partial}{\partial s}|_{s=s_0} \exp^{-1}_{c(s_0)} \gamma_{v(s)}(\rho(s_0))|| = ||Y_N(\rho(s_0); v(s_0))||. \tag{2.13}$$

It follows from (2.9), (2.10), (2.12) and (2.13) that we get (2.5). □

**Theorem 2.3.** *For each cut point $q$ of $N$ which is not a focal point of $N$ along each $N$-segment reaching $q$, the space of directions at $q$ coincides with the cut locus of $\Lambda_N(q)$ in the sphere $S_qM$. Here the space of directions at $q$ is defined to be the set of all limit unit tangent vectors at $q$ of sequences $\{v_q(q_i)\}$ as cut points $q_i$ of $N$ tend to $q$.*

*Proof.* By Proposition 2.2, we have proved that any element of the space of directions at $q$ is a cut point of $\Lambda_N(q)$. Suppose that there exists a cut point $v$ of $\Lambda_N(q)$ that is not an element of the space of directions at $q$. Since $v$ is a cut point of $\Lambda_N(q)$, we may choose two unit speed geodesics $c_i : [0, \theta] \to S_qM$, $i = 1, 2$, joining $v_i$ to $v$, none of which meet $\Lambda_N(q)$, except $v_i$. For each positive $\epsilon$ let $\gamma_\epsilon : [0, 2\theta] \to M$ be a curve joining $\exp(\epsilon v_1)$ to $\exp(\epsilon v_2)$ such that $\gamma_\epsilon(t) = \exp(\epsilon c_1(t))$ and $\gamma_\epsilon(t) = \exp(\epsilon c_2(2\theta - t))$ for $t \in [0, \theta]$ and $t \in [\theta, 2\theta]$, respectively. By definition, for any sufficiently small positive $\epsilon$, the curve $\gamma_\epsilon$ does not meet the cut locus of $N$. Thus, there exists a unique curve $\tilde\gamma_\epsilon : [0, 2\theta] \to \nu$ in the open subset $\{tv;\, 0 < t < \rho(v),\, v \in U\nu\}$ of the normal bundle that satisfies $\exp^\perp(\tilde\gamma_\epsilon(t)) = \gamma_\epsilon(t)$. It is clear that a family of curves $\{\tilde\gamma_\epsilon(t)\}_\epsilon$ is equicontinuous, since the lengths of the velocity vectors of $\tilde\gamma_\epsilon$ are bounded. It follows from the Ascoli-Arzela theorem that the family has a limit curve $\tilde\gamma$, which is continuous, as $\epsilon$ goes to zero. Hence, $\exp^\perp(\tilde\gamma(t)) = q$ for any $t \in [0, 2\theta]$. If we define a continuous curve $\xi(t)$ in $U\nu$ by

$$\xi(t) := \frac{1}{||\tilde\gamma(t)||}\tilde\gamma(t),$$

then from the construction it follows that $\rho(\xi(t)) \geq ||\tilde{\gamma}(t)||$. Thus $||\tilde{\gamma}(t)|| = d(N, q)$ for any $t \in [0, 2\theta]$, since $\exp^{\perp}(\tilde{\gamma}(t)) = q$. Therefore we get a family of $N$-segments $\{\gamma_{\xi(t)}[0, d(N,q)]\}_{t\in[0,2\theta]}$ reaching $q$ such that

$$\acute{\gamma}_{\xi(0)}(d(N,q)) = -v_1 \quad \text{and} \quad \acute{\gamma}_{\xi(2\theta)}(d(N,q)) = -v_2.$$

This implies $q$ is a focal point of $N$, which contradicts the hypothesis of the theorem. Thus, the proof is complete. □

To prove the local Lipschitz continuity of $\rho$ at $v_0$, fix any $v_0 \in U_p\nu$ with $\rho(v_0) < \infty$. Let $B_D(v_0; \delta_0(v_0))$ denote a relatively compact convex ball in $(U\nu, G)$ centered at $v_0$ with radius $\delta_0(v_0)$, on which $\rho \leq \rho(v_0) + 1$.

**Lemma 2.4.** *There exist positive numbers $C_1(v_0)$ and $\delta_1(v_0)$ $(< \delta_0(v_0))$ such that for any $v, w \in B_D(v_0; \delta_1(v_0))$ with $\gamma_v(\rho(v)) = \gamma_w(\rho(w))$ the inequality*

$$C_1(v_0)D(v,w) < \angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w))) \tag{2.14}$$

*holds.*

*Proof.* Since $\gamma_{v_0}(t_0)$, where $t_0 = \frac{\rho(v_0)}{2}$, is not a focal point of $N$ along $\gamma_{v_0}$, the differential of the normal exponential map has maximal rank at $t_0v_0$. Thus, there exist a positive constant $C_1$ and a convex ball $B_D(v_0; \delta_1(v_0))$ $(\delta_1(v_0) < \delta_0(v_0))$ in $U\nu$ such that

$$C_1 \ D(u,w) < d(\gamma_u(t_0), \gamma_w(t_0)) \tag{2.15}$$

for any $u, w \in B_D(v_0; \delta_1(v_0))$. By taking a smaller $\delta_1(v_0)$, we may assume that

$$\frac{3}{2}t_0 < \rho(v) < \frac{5}{2}t_0$$

on $B_D(v_0; \delta_1(v_0))$. Let $K$ be the closure of the set $\{\exp(\rho(v)v); v \in B_D(v_0; \delta_1(v_0))\}$. Note that $K$ is compact, because $\rho < \frac{5}{2}t_0$ on $B_D(v_0; \delta_1(v_0))$. Thus, there exists a constant $C_2$ such that

$$\max\{||Y(t)||; 0 \leq t \leq 2t_0\} \leq C_2$$

for any Jacobi field $Y$ along a geodesic that emanates from $K$ with initial conditions $Y(0) = 0, ||Y'(0)|| = 1$. Suppose that $v, w \in B_D(v_0; \delta_1(v_0))$ satisfy $\gamma_w(\rho(w)) = \gamma_v(\rho(v)) =: q$. Let $\xi : [0, \phi] \to S_qM$ denote a unit speed minimal geodesic joining $-\dot{\gamma}_v(\rho(v))$ to $-\dot{\gamma}_w(\rho(w))$, where $\phi = \angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w)))$. The curve $x(\theta) = \exp_q(t_1\xi(\theta)), \theta \in [0, \phi]$, joins $\gamma_v(t_0)$ to $\gamma_w(t_0)$, where $t_1 := \rho(w) - t_0 = \rho(v) - t_0$. By definition,

$$d(\gamma_v(t_0), \gamma_w(t_0)) \leq \int_0^{\phi} ||\dot{x}(\theta)||d\theta.$$

Since $||\dot{x}(\theta)|| \leq C_2$, we get

$$d(\gamma_v(t_0), \gamma_w(t_0)) \leq C_2\phi = C_2\angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w))). \tag{2.16}$$

From (2.15) and (2.16) we get (2.14). □

Since the differential of the map $(\pi, \exp) : TM \to M \times M, (q, v) \to (q, \exp_q(v))$ has maximal rank at each zero vector, it has a $C^{\infty}$ local inverse $\Phi$ on an open set $U_r \supset \{(\gamma_{v_0}(t), \gamma_{v_0}(t)); 0 \leq t \leq r\}$, where $r := \rho(v_0) + 1$. Choose a positive number $\delta_2(v_0)$ $(< \delta_1(v_0))$ such that, for any $v_1, v_2 \in B_D(v_0; \delta_2(v_0))$ and any $t \in [0, r]$, $(\gamma_{v_1}(t), \gamma_{v_2}(t)) \in U_r$.

**Definition 2.7.** For each distinct $v, \tilde{v} \in B_D(v_0; \delta_2(v_0))$ let $X(t; v, \tilde{v})$ denote the vector field along $\gamma_v|_{[0,r]}$ defined by

$$X(t; v, \tilde{v}) := \frac{1}{\psi}\Phi(\gamma_v(t), \gamma_{\tilde{v}}(t)),$$

where $\psi = D(v, \tilde{v})$.

It is trivial that there exists a positive constant $C_2(v_0)$ such that

$$\angle(\dot{\gamma}_v(\rho(v)), \dot{\gamma}_w(\rho(w))) \geq C_2(v_0) \tag{2.17}$$

for any $v \in B_D(v_0; \delta_3(v_0))$ and $w \in U\nu \setminus B_D(v_0; \delta_1(v_0))$ with $\gamma_v(\rho(v)) = \gamma_w(\rho(w))$, where $\delta_3(v_0) := \frac{\delta_2(v_0)}{2}$.

**Lemma 2.5.** *There exists a positive number $C_3(v_0)$ such that for any $t \in [0, r]$ and any unit speed minimizing geodesic $\xi(s)$ $(0 \leq s \leq \psi)$ in $B_D(v_0; \delta_2(v_0))$*

(2.18)
$$||X(t; \xi(0), \xi(\psi)) - Y_N(t; \xi(0))|| + ||X'(t; \xi(0), \xi(\psi)) - Y'_N(t; \xi(0))|| \leq C_3(v_0)\psi,$$

*where $Y_N(t; \xi(0))$ is the $N$-Jacobi field along $\gamma_{v(0)}$ defined in Definition 2.4.*

*Proof.* Since $(\gamma_{\xi(0)}(t), \gamma_{\xi(s)}(t)) \in U_r$ for any $t \in [0, r]$ and any $s \in [0, \psi]$, the vector field $\Phi(\gamma_{\xi(0)}(t), \gamma_{\xi(s)}(t))$ along $\gamma_{\xi(0)}|_{[0,r]}$ is well-defined for each $s \in [0, \psi]$. Let $f : [0, r] \times B_D(v_0; \delta_2(v_0)) \times B_D(v_0; \delta_2(v_0)) \to TM$ be a $C^\infty$ map defined by

$$f(t, v_1, v_2) := \Phi(\gamma_{v_1}(t), \gamma_{v_2}(t))$$

and put $h(s) := f(t, \xi(0), \xi(s))$. Since

$$h(\psi) = h'(0)\psi + \psi^2 \int_0^1 u \int_0^1 h''(su\psi)ds\, du$$

and $h'(0) = Y_N(t; \xi(0))$, we get

$$X(t; \xi(0), \xi(\psi)) = Y_N(t; \xi(0)) + \psi \int_0^1 u \int_0^1 h''(su\psi)ds\, du.$$

Hence, the inequality (2.18) is trivial. $\square$

**Lemma 2.6.** *Let $v : (a, b) \to B_D(v_0; \delta_3(v_0))$ be a unit speed geodesic such that $\lambda(s) > \rho(s)$ on $(a, b)$. Then for each $s \in (a, b)$,*

$$|D^{\pm}\rho(s)| \leq C(J_N) \max\left(\cot \frac{C_4(v_0)}{2}, \frac{\pi^2 C_3(v_0)C_1(v_0)^{-2}}{2}\right), \tag{2.19}$$

*where*

$$C_4(v_0) = \min(C_2(v_0), C_1(v_0)\delta_3(v_0))$$

$$C(J_N) = \sup\{||Y_N(t; v(s))||, ||Y'_N(t; v(s))||; 0 \leq t \leq r,$$
$$v(s) \textit{ is a unit speed geodesic in } B_D(v_0; \delta_3(v_0))\}.$$

*Proof.* Let $e_3(s)$ denote the unit tangent vector satisfying

$$w_+(s) = e_1(s)\cos\phi(s) + e_3(s)\sin\phi(s), \tag{2.20}$$

where $\phi(s) := \angle(w_+(s), e_1(s))$. Since $\angle(e_1(s), \xi_+(s)) = \theta_+(s)$ and $\angle(\xi_+(s), e_2(s)) < \frac{\pi}{2}$, it follows that

$$\xi_+(s) = e_1(s)\cos\theta_+(s) + e_2(s)\sin\theta_+(s). \tag{2.21}$$

Since $\xi_+(s)$ is orthogonal to $w_+(s) - e_1(s)$, it follows from (2.20) and (2.21) that

$$\cot \theta_+(s) = \frac{\sin \phi(s)}{1 - \cos \phi(s)} g(e_2(s), e_3(s)). \tag{2.22}$$

Hence, by (2.5), we get

(2.23)
$$D^+\rho(s) = -\frac{\sin \phi(s)}{1 - \cos \phi(s)} g(Y_N(\rho(s)), e_3(s)) = -\cot \frac{\phi(s)}{2}\ g(Y_N(\rho(s)), e_3(s)),$$

where $Y_N(t) := Y_N(t; v(s))$. Let $\tilde{v}(s) \in U\nu$ denote the vector satisfying $-w_+(s) = \dot{\gamma}_{\tilde{v}(s)}(\rho(\tilde{v}(s)))$. If $\phi(s)$ is not less than $C_4(v_0)$, then from (2.23) it is trivial that

$$|D^+\rho(s)| \leq C(J_N)\ \cot \frac{C_4(v_0)}{2}. \tag{2.24}$$

If $\phi(s)$ is less than $C_4(v_0)$, then it follows from (2.14) and (2.17) that $D(v(s), \tilde{v}(s)) < \delta_3(v_0)$. Thus, by the triangle inequality, $\tilde{v}(s) \in B_D(v_0; \delta_2(v_0))$. The vector field $X(t) := X(t; v(s), \tilde{v}(s))$ is well-defined by Definition 2.7. Since $X(\rho(s)) = 0$, we get

(2.25)
$$X'(\rho(s)) = \frac{1}{\psi(s)}(e_1(s) - w_+(s)) = \frac{1}{\psi(s)}((1 - \cos \phi(s))e_1(s) - e_3(s) \sin \phi(s)),$$

where $\psi(s) := D(v(s), \tilde{v}(s))$. Let $\xi : [0, \psi(s)] \to B_D(v_0; \delta_2(v_0))$ denote the unit speed minimal geodesic joining $v(s)$ to $\tilde{v}(s)$. It follows from (2.23) and (2.25) that

$$D^+\rho(s) = \cot \frac{\phi(s)}{2}\ \frac{\psi(s)}{\sin \phi(s)}\ g(Y_N(\rho(s)), X'(\rho(s))). \tag{2.26}$$

It follows from (1.3) that

$$\begin{aligned} g(Y_N(\rho(s)), X'(\rho(s))) = g(Y_N(\rho(s)), X'(\rho(s)) - X_N'(\rho(s))) \\ + g(Y_N'(\rho(s)), X_N(\rho(s))), \end{aligned}$$

where $X_N(t) := Y_N(t; \xi(0))$. Hence, by (2.26), we have

(2.27)
$$|D^+\rho(s)| \leq \cot \frac{\phi(s)}{2}\ \frac{\psi(s)}{\sin \phi(s)}\ C(J_N)(||X'(\rho(s)) - X_N'(\rho(s))|| + ||X_N(\rho(s))||).$$

Since $X(\rho(s)) = 0$, by (2.14), (2.18) and (2.27), we get

$$|D^+\rho(s)| \leq \cot \frac{\phi(s)}{2}\ \frac{\psi(s)^2}{\sin \phi(s)} C(J_N) C_3(v_0) \leq \frac{\pi^2}{2} C_1(v_0)^{-2} C_3(v_0) C(J_N). \tag{2.28}$$

By (2.24) and (2.28), we get (2.19). The estimate for $D^-\rho(s)$ is the same as the one for $D^+\rho(s)$. □

*Proof of Theorem B.* Let $v_0 \in U\nu$ be any vector with $\rho(v_0) < \infty$. Choose a small convex ball $B_D(v_0; \delta_4(v_0))$, $\delta_4(v_0) < \delta_3(v_0)$, on which $\rho < \lambda_1$ or $\lambda_1$ is Lipschitz continuous with Lipschitz constant $L(\lambda_1)$. Let $v_1, v_2 \in B_D(v_0; \delta_4(v_0))$ be any distinct vectors with $\rho(v_1) \leq \rho(v_2)$. Let $\xi : [0, \psi] \to B_D(v_0; \delta_4(v_0))$ be the unit speed geodesic joining $v_1$ to $v_2$, so that $\psi = D(v_1, v_2)$. If $\lambda_1(v_1) = \rho(v_1)$, then

$$|\rho(v_1) - \rho(v_2)| = \rho(v_2) - \rho(v_1) \leq \lambda_1(v_2) - \lambda_1(v_1) \leq L(\lambda_1) D(v_1, v_2). \tag{2.29}$$

Suppose that $\lambda_1(v_1) > \rho(v_1)$. Let $(0, a)$ be the maximal open subinterval of $[0, \psi]$ on which $\lambda_1 > \rho$. By Lemma 2.5,

$$|D^{\pm}\rho(s)| \leq C_5(J_N, v_0)$$

on $(0, a)$, where

$$C_5(J_N, v_0) := C(J_N) \max\left(\cot \frac{C_4(v_0)}{2}, \frac{\pi^2 C_3(v_0) C_1(v_0)^{-2}}{2}\right).$$

Hence, $\rho \circ \xi$ is Lipschitz continuous with Lipschitz constant $C_5(J_N, v_0)$ on $[0, a]$. In particular,

$$|\rho(v_1) - \rho(\xi(a))| \leq C_5(J_N, v_0)a. \tag{2.30}$$

If $a < \psi$, then $\lambda_1(\xi(a)) = \rho(\xi(a))$. Thus by (2.30), we get

$$|\rho(v_1) - \rho(v_2)| \leq \lambda_1(v_2) - \lambda_1(\xi(a)) + |\rho(\xi(a)) - \rho(v_1)| \leq L(\rho)D(v_1, v_2), \tag{2.31}$$

where $L(\rho) := \max(L(\lambda_1), C_5(J_N, v_0))$. If $a = \psi$, then (2.31) is trivial by (2.30). Therefore, by (2.29) and (2.31),

$$|\rho(v_1) - \rho(v_2)| \leq L(\rho)D(v_1, v_2)$$

for any $v_1, v_2 \in B_D(v_0; \delta_4(v_0))$. □

The length $L(c)$ of a continuous curve $c : [a, b] \to M$ is defined as

$$L(c) := \sup \sum_{i=1}^{k} d(c(t_{i-1}), c(t_i)),$$

where the supremum is taken over all subdivisions

$$a = t_0 < t_1 < \cdots < t_k = b$$

of $[a, b]$. Note that any absolutely continuous curve has finite length (cf. [28] for the definition of an absolutely continuous curve). We omit the proof of the following lemma, since it is standard (cf. [28]).

**Lemma 2.7.** *For any absolutely continuous curve* $c : [a, b] \to M$,

$$L(c) = \int_a^b ||\dot{c}(t)||\, dt.$$

We introduce an interior metric $\delta$ on a component $C_N^0$ of $C_N$ by

$$\delta(p, q) := \inf\{L(c); c \text{ is a continuous curve on } C_N^0 \text{ joining } p \text{ to } q\}.$$

By Theorem B, $\delta(p, q)$ is finite for any $p, q \in C_N^0$. Any two points $p, q \in C_N^0$ can be connected by a minimal curve $c$; that is, there exists a continuous curve $c$ joining $p$ to $q$ such that $\delta(p, q) = L(c)$ (for example, cf. Theorem 5.18 in [3]). It follows from Lemma 2.7 that $\delta$ coincides with the usual definition of the Riemannian distance function, or, in other words,

$$\delta(p, q) = \inf\{\int_0^1 ||\dot{c}(t)|| dt; c \text{ is an absolutely continuous}$$
$$\text{curve on } C_N^0 \text{ joining } p \text{ to } q\}.$$

*Proof of Corollary C.* Let $\{p_n\}$ be a sequence of points in $C_N^0$ such that

$$\lim_{n\to\infty} d(p, p_n) = 0.$$

Since the cut locus is closed, $p$ is a cut point of $N$. For each $p_n$ choose a vector $v_n \in U\nu$ with $\exp(\rho(v_n)v_n) = p_n$. Let $v \in U\nu$ be a limit vector of the sequence $\{v_n\}$. Let $\xi_n : [0, D(v, v_n)] \to U\nu$ be a minimizing geodesic joining $v$ to $v_n$, and put $\overline{\xi}_n(t) := \exp(\rho(\xi_n(t))\xi_n(t))$. Since $\overline{\xi}_n$ is a (Lipschitz) continuous curve in $C_N^0$ joining $p$ to $p_n$, we get

$$\delta(p, p_n) \leq L(\overline{\xi}_n). \tag{2.32}$$

Since $\rho$ is locally Lipschitz, the map $w \in U\nu \to \exp(\rho(w)w) \in M$ is also locally Lipschitz. Thus, there exist a positive constant $C$ and a neighborhood $V$ around $v$ such that

$$L(\overline{\xi}_n) \leq CL(\xi_n) = CD(v, v_n) \tag{2.33}$$

for any $n$ with $v_n \in V$. By (2.35) and (2.36), we get $\lim_{n\to\infty} \delta(p, p_n) = 0$. Thus, the topology introduced from $\delta$ coincides with the relative topology of $(M, g)$. The other claims are clear from this property. □

## 3. Open problems and examples

The functions $\lambda_k$ are not always differentiable, except when $M$ is of dimension 2. The following example shows that $\lambda_1$ need not be differentiable.

**Example 3.1.** Let $M$ denote the Riemannian product of two 2-dimensional unit spheres $S^2$. Choose a unit tangent vector $v_1$ to $S^2$ at a point $p_1$. For each $\theta \in [0, \frac{\pi}{2}]$, we define a geodesic $\gamma_\theta$ on $M$ by

$$\gamma_\theta(t) := (\exp(tv_1 \cos\theta), \exp(tv_1 \sin\theta)).$$

Let $\lambda_1$ denote the distance function to the first conjugate tangent vectors of the point $p := (p_1, p_1) \in M$. Thus

$$\lambda_1(\dot\gamma_\theta(0)) = \min\left(\frac{\pi}{\cos\theta}, \frac{\pi}{\sin\theta}\right).$$

Hence $\lambda_1(\dot\gamma_\theta(0))$ is not differentiable at $\theta = \frac{\pi}{4}$, that is, $\lambda_1$ is not differentiable at $\left(v_1/\sqrt{2}, v_1/\sqrt{2}\right)$.

There exist many surfaces admitting a cut locus with branch points (for example cf. [7] or the following example). This implies such a cut locus need not have curvature bounded below in the sense of Alexandrov.

**Example 3.2.** Let $N$ be a smooth convex Jordan curve in the 2-dimensional Euclidean plane $\mathbf{R}^2$ which contains a regular triangle $T$, except around its three vertices. Then the cut locus of $N$ contains three line segments emanating from the center of $T$.

The following example shows that there is a cut locus containing a neighborhood of the vertex of a flat cone. This implies this cut locus cannot have curvature bounded above in the sense of Alexandrov.

**Example 3.3.** Take a $C^\infty$ Jordan arc $\mathcal{C}$ in the $yz$ plane in the 3-dimensional Euclidean space $\mathbf{R}^3$ with endpoints $(0,0,\pm 1)$ as follows:

(1) $\mathcal{C}$ contains three arcs

$$C_1 := \{(0, \cos\theta, \sin\theta); -\frac{\pi}{2} \le \theta \le -\frac{\pi}{2} + \delta\},$$

$$C_2 := \{(0, \cos\theta, \sin\theta); -\frac{\pi}{4} + \delta \le \theta \le \frac{\pi}{2}\}$$

and

$$C_3 := \{(0, \frac{1}{\sqrt{2}} + \frac{\delta}{10}\cos\phi, -\frac{1}{\sqrt{2}} + \frac{\delta}{10}\sin\phi); -\frac{\pi}{4} - \delta \le \phi \le -\frac{\pi}{4} + \delta\},$$

where $\delta$ is a sufficiently small positive constant.

(2) $\mathcal{C} \setminus (C_1 \cup C_2 \cup C_3)$ consists of two Jordan subarcs which are mutually symmetric with respect to the line through $(0,0,0)$ and $(0,1,-1)$.

(3) The cut locus of $\mathcal{C} \setminus \{(0,0,\pm 1)\}$ in the $yz$ plane is the line segment with endpoints $(0,0,0)$ and $(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ .

Let $N$ be the surface of revolution obtained by rotating $\mathcal{C}$ about the $z$ axis. Then the cut locus of $N$ coincides with a cone

$$\{(x,y,z); x^2 + y^2 = z^2,\ -\frac{1}{\sqrt{2}} \le z \le 0\}.$$

The cut loci constructed in Examples 3.2 and 3.3 are those of a submanifold that is not a single point. By making use of Weinstein's technique ([26]), we may regard these cut loci as being of a single point.

Finally, we state five interesting open problems, some of which might be proved using the local Lipschitz continuity of the function $\rho$.

J. Hebda and J. Itoh affirmatively solved Ambrose's problem in the 2-dimensional case (cf. [11], [13]). They solved it by proving that the cut locus of a point on a 2-dimensional Riemannian manifold has finite 1-dimensional Hausdorff measure. Hebda had pointed out in [10] that it is sufficient to prove the property above to solve the problem in the 2-dimensional case. Theorem B generalizes this property for any dimensional compact Riemannian manifolds. Thus we might be able to solve Ambrose's problem using this property.

**Problem 3.1.** *Solve Ambrose's problem for any dimensional Riemannian manifold.*

The authors proved in [14] that for each cut point $q$ of a point $p$ on $M$, there exists a nonnegative integer $k$ such that the cut locus of $p$ is locally $k$-dimensional around $q$. We call the integer $k$ the *local dimension* of the cut locus at $q$.

**Problem 3.2.** *Let $q$ denote a cut point of a point $p$ on $M$ at which the local dimension of the cut locus is $k$. Is the cut locus locally a $k$-dimensional submanifold of $M$ around $q$, except for a $k$-null subset of $M$? Here a subset of $M$ is said to be $k$-null if it is of $k$-dimensional Hausdorff measure zero.*

Hereafter $N$ denotes an embedded submanifold of a complete Riemannian manifold $M$. A point $q \in M \setminus N$ is called a *critical point* of the distance function from $N$ if for each unit tangent vector $v$ at $q$ there exists a unit tangent vector $w$ in $\Lambda_N(q)$ such that the angle made by $v$ and $w$ is not greater than $\frac{\pi}{2}$. A real number $c$ is called a *critical value* of the distance function from $N$ if there exists a critical point

$q$ whose distance is $c$ from $N$. It is well-known that for each positive number $c$ the set of all points whose distances are $c$ is a topological hypersurface in $M$, if $c$ is not a critical value of the distance function (cf. [5]). In [22], it was proved that the set of all critical values of the distance function from a compact subset in an Alexandrov surface is of Lebesgue measure zero. Does what we call a "Sard Theorem for the distance function" hold for the distance function from $N$? Namely,

**Problem 3.3.** *Is the set of all critical values of the distance function from $N$ of Lebesgue measure zero?*

We showed in Examples 3.2 and 3.3 that the cut locus is not always an Alexandrov space. How about the tangent cut locus?

**Problem 3.4.** *Is the tangent cut locus of $N$ an Alexandrov space?*

We proved in Theorem 2.3 that the space of directions at a non-focal cut point $q$ of $N$ coincides with the cut locus of $\Lambda_N(q)$ in $S_qM$. Here a *non-focal* cut point $q$ is a cut point that is not a focal point along each $N$-segment reaching $q$. Therefore, the following problem is an interesting investigatation into the structure of a cut locus.

**Problem 3.5.** *Let $q$ be a non-focal cut point of $N$. Then, is $S(q;\delta)\cap C_N$ homeomorphic to the cut locus of $\Lambda_N(q)$ in $S_qM$ for any sufficiently small positive $\delta$? Here $S(q;\delta)$ denotes a geodesic sphere in $M$ centered at $q$ with radius $\delta$.*

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